Critical behaviour of a long-range non-equilibrium system



1994 J. Phys. A: Math. Gen. 27 3389

(http://iopscience.iop.org/0305-4470/27/10/014)

J. Phys. A: Math. Gen. 27 (1994) 3389–3395. Printed in the UK

# Critical behaviour of a long-range non-equilibrium system

M C Marques† and A L Ferreira‡

† Departamento de Física da Universidade do Porto, Praça Gomes Teixeira, 4000 Porto, Portugal
‡ Departamento de Física da Universidade de Aveiro, 3800 Aveiro, Portugal

Received 5 July 1993

**Abstract.** The critical behaviour of a long-range, one-dimensional non-equilibrium system where the desorption rate decreases with distance as some power $\sim r^{-\alpha-d}$ is studied by means of the coherent anomaly method. The $\alpha$-dependence of the critical parameter $\lambda_c$ and the critical exponent $\beta$ is determined.

## 1. Introduction

The simplest examples of non-equilibrium critical systems are lattice models where a binary site variable evolves according to certain dynamic rules depending on the state of the sites closely neighbouring a given site. Models like the contact process, the A-model and variations of these [1], arise in fields such as the propagation of epidemics, the spreading of a liquid through porous media, and processes of adsorption–desorption of particles on a surface; they all exhibit phase transitions into an absorbing state and belong to the universality class of directed percolation.

A variety of methods have been used to study these systems, ranging from cluster mean-field theories, MC simulations and series expansions [1] to RG methods [2] and $\epsilon$ expansions in reggeon field theories [4, 5]. The coherent anomaly method (CAM) was also applied to these systems [6, 7]; more recently [8] an implementation of CAM by means of a series of cluster approximations (which improve on the previous ones by appropriately considering higher degree correlations) led to rather good estimates of the critical parameter and critical exponents for the one-dimensional A-model.

All the models referred to above have in common the fact that they obey dynamic rules which are local, i.e. only nearest neighbours are envolved. However a more realistic model for the spreading of an infection is to consider an infection probability which decreases with distance like some power $r^{-\alpha-d}$, as proposed by Grassberger [5]. The dynamic rules for such model are no longer local which increases enormously the computer time required for numerical simulations, as seen in other systems with long-range interactions [9]. The difficulties envisaged have, naturally, enhibited the investigation of these long-range effects and the only results we are aware of are the ones obtained by Grassberger himself using $\epsilon$-expansion expressions for the critical exponents (with $\epsilon = d_c - d$, where $d_c = 2\alpha$ is the upper critical dimension). The investigation of non-equilibrium phase transitions faces aditional difficulties as compared with the case of equilibrium systems, which have therefore been more widely studied; this happens with the one-dimensional long-range Ising system, for which a few exact results are known [3].

In this paper we consider a long-range, non-equilibrium system in one dimension and study it by the coherent anomaly method. A similar study of the long-range one-dimensional

0305-4470/94/103389+07$19.50  

Ising system has been performed by Monroe *et al* [10]. In both cases the method relies on a comparison of results for a sequence of mean-field like approximations: these are easily constructed for the latter, where the equilibrium probability distribution relates to the Ising Hamiltonian; in the former case, one studies the stationary solutions of the master equation for the time evolution of the probability of $L$-site clusters configurations, in the manner used before for the short-range case [8]. A set of different values of $\alpha$ between 0 and 2 is considered allowing for an investigation of the $\alpha$-dependence of critical parameters and exponents. This dependence is then compared with the field-theory predictions according to which mean-field like exponents are to be expected for $0 < \alpha < 0.5$ and short-range (contact-process) critical behaviour is predicted for $\alpha \geqslant 2$.

In section 2 the dynamic rules for the system are presented together with the method of study and the sequence of cluster mean-field approximations. In section 3 the CAM estimates for the critical parameter and the critical exponent $\beta$ are displayed; the results shown in subsection 3.2 were obtained by using higher-degree mean-field like approximations (in the spirit of [8]). The computer time required in that case was considerably higher than that needed for the approximations considered in subsection 3.1. In section 4 results of a computer simulation for a system of fixed size and a single value of $\alpha$ are presented and compared to the CAM estimates of subsection 3.1. Finally, section 5 contains a discussion of the results and some concluding remarks.

## 2. The model and the method of study

### *2.1. The model*

We consider a 1D array of sites in which a variable $\sigma_i$ takes the values 0 or 1 , corresponding to site $i$ being vacant or occupied by a particle. The rate of deposition is $\lambda$ (provided the site where deposition is attempted is found to be vacant) and the rate of desorption is

$$w(\sigma_i = 1 \to \sigma_i = 0) = \sum_{j \neq i} (1 - \sigma_j) \frac{1}{|i - j|^{1+\alpha}} .$$

The rate of desorption is then dependent on the number of vacant sites and their respective distance to the chosen site. One can see that considering only the nearest-neighbour terms ($j = i - 1$ and $j = i + 1$) in the summand just corresponds to the desorption rates for the contact process: rate 2 if both neighbours are vacant, rate 1 if only one neighbour is vacant and no desorption if both are occupied. Of course, one expects the short-range case to be recovered for a sharp decrease with distance ($\alpha \geqslant 2$) of the other terms in the summand.

### *2.2. The method*

Suzuki's coherent anomaly method [11] (CAM) relies on a sequence of mean-field like approximations obtained self-consistently with clusters of increasing size $L$. In the present non-equilibrium system, the order parameter of the transition is the fraction of vacant sites, $\rho$, which is zero above a certain critical value $\lambda_c$ and behaves like $\rho \sim (\lambda_c - \lambda)^\beta$ for $\lambda \leqslant \lambda_c$. Within the $L$-size cluster approximation the order parameter $\rho_L$ vanishes at $\lambda_c^L$ as $\rho_L \sim A_L(\lambda_c^L - \lambda)^{\beta_{MF}}$, where $\beta_{MF} = 1$ is the mean-field exponent. The basic assumption of CAM relates the size dependence of the amplitude $A_L$ with the true exponent $\beta$, in the following manner:

$$A_L \sim (\lambda_c^L - \lambda_c)^{\beta - \beta_{MF}}$$

or, with the inclusion of first-order corrections

$$A_L = a(\lambda_c^L - \lambda_c)^{\beta-\beta_{MF}} + b(\lambda_c^L - \lambda_c)^{\beta-\beta_{MF}+1} \tag{1}$$

and the critical parameter value, $\lambda_c$, can be obtained by a fitting of the form

$$\lambda_c^L = \lambda_c + a'L^{-\omega} + b'L^{-x} \tag{2}$$

where the last term is a correction to the leading $L^{-\omega}$ term. In a manner typical of finite systems, $\omega$ is usually the inverse of $\nu_\perp$, the critical exponent of the correlation length [11, 14]; however, this also depends on the nature of the approximation [12], as seems to be the case in the series of approximations described in subsection 2.3.1. The underlying relation between $\omega$ and $1/\nu_\perp$ still needs to be clarified.

### *2.3. The cluster mean-field approximations*

*2.3.1. Approximation A.* The self-consistent approximations which are most easily implemented in this system are obtained in the following way:

A master equation for the time evolution of the probability of a $L$(odd)-site cluster $P_L(\sigma_1, ..., , \sigma_L, t)$ is written, where the contribution for the desorption rates coming from the sites inside the cluster are treated exactly, whereas the probability of any site outside the cluster to be vacant is set equal to $m$, independently of the correlations with its neighbours. We then look at stationarity and impose selfconsistently that the average value of the central-site variable must equal $m$. The equation is then solved numerically and the values of $\lambda_c^L$ and $A_L$ are obtained.

*2.3.2. Approximation B.* We have also considered another sucession of higher-degree, mean-field approximations (which, however, take much more computer time to be implemented). They improve on the previous ones in the sense that the correlations in the boundaries of the clusters are more appropriately treated. The master equation for $P_L(\sigma_1, ..., \sigma_L, t)$ depends upon the conditional probabilities $P(\sigma_j|\sigma_1, ..., \sigma_L, t)$, where $j$ denotes a site outside of the cluster. Approximation A (subsection 2.3.1) corresponds to take this conditional probability equal to the average $\sigma$ value of the central site. A more complete approximation can however be used if we estimate the conditional probability from $P_L(\sigma_1, ..., \sigma_L, t)$ in the spirit of what was done in [8] and [13]. The approximation is thus improved for the conditional probabilities of sites $j$ whose distance to the end sites of the cluster is less than $L$ ($|j-1| < L$ or $|j-L| < L$). For more distant sites there is no way of including the correlations with the cluster sites, even approximately, and approximation (subsection 2.3.1) is used with $m$ self-consistently equal to the average fraction of vacant sites, $\rho(t)$.

## 3. Results of CAM

### *3.1. Results obtained from approximation A*

In table 1 and table 2 we list the values of $\lambda_c^L$ and $A_L$ obtained for the different values of $\alpha$. They were obtained by numerically solving the self-consistent equations of 2.3.1 and fitting pairs of $(\lambda, \rho)$ values (very close to the transition) to the form

$$\rho = A_L(\lambda_c^L - \lambda) + b(\lambda_c^L - \lambda)^2 + c(\lambda_c^L - \lambda)^3 .$$

**Table 1.** The parameter $\lambda_c^L$ as obtained for different cluster sizes and different values of $\alpha$, within approximation A (subsection 2.3.1).

| | $\alpha$ | | | | |
|---|---|---|---|---|---|
| $L$ | 0.25 | 0.5 | 0.75 | 1.0 | 2.0 |
| 3 | 8.537 8123 | 4.874 6367 | 3.517 6144 | 2.826 5817 | 1.820 9414 |
| 5 | 8.470 0124 | 4.759 5828 | 3.367 1712 | 2.651 4345 | 1.606 1686 |
| 7 | 8.434 7714 | 4.697 2635 | 3.284 1988 | 2.554 3626 | 1.489 7717 |
| 9 | 8.412 7116 | 4.657 1751 | 3.230 2286 | 2.491 0826 | 1.415 1884 |
| 11 | 8.397 3991 | 4.628 7842 | 3.191 7143 | 2.445 8819 | 1.362 6436 |
| 13 | 8.386 0452 | 4.607 3993 | 3.162 5417 | 2.411 6353 | 1.323 2852 |

**Table 2.** The parameter $A_L$ as obtained for different cluster sizes and different values of $\alpha$, within approximation A (subsection 2.3.1).

| | $\alpha$ | | | | |
|---|---|---|---|---|---|
| $L$ | 0.25 | 0.5 | 0.75 | 1.0 | 2.0 |
| 3 | 0.120 032 | 0.217 895 | 0.313 088 | 0.402 379 | 0.678 731 |
| 5 | 0.122 230 | 0.230 056 | 0.345 277 | 0.463 151 | 0.880 362 |
| 7 | 0.123 481 | 0.237 847 | 0.367 695 | 0.508 167 | 1.048 958 |
| 9 | 0.124 319 | 0.243 509 | 0.384 959 | 0.544 362 | 1.196 256 |
| 11 | 0.124 933 | 0.247 925 | 0.399 038 | 0.574 860 | 1.328 408 |
| 13 | 0.125 409 | 0.251 529 | 0.410 947 | 0.601 331 | 1.449 102 |

**Table 3.** $\lambda_c$ and $w$ as obtained from fitting (2) with $x = 2$, and using approximation A (subsection 2.3.1): $\lambda_c^L = \lambda_c + aL^{-w} + bL^{-2}$, for different values of $\alpha$. The estimated errors for $\lambda_c$ and $w$ are respectively 0.002 and 0.01.

| $\alpha$ | $\lambda_c$ | $w$ |
|---|---|---|
| 0.25 | 8.287 | 0.66 |
| 0.5 | 4.395 | 0.59 |
| 0.75 | 2.862 | 0.57 |
| 1.0 | 2.060 | 0.57 |
| 2.0 | 0.965 | 0.64 |

A fit of the form (2) was then done in order to determine $\lambda_c$, for the different values of $\alpha$. Better results were obtained using a constant value of $x = 2$, thus reducing the number of

**Table 4.** $\beta(\alpha)$ as obtained from the fitting: $A_L = a(\lambda_c^L - \lambda_c)^{\beta-1} + b(\lambda_c^L - \lambda_c)^{\beta} + c(\lambda_c^L - \lambda_c)^{\beta+1}$, and using approximation A (subsection 2.3.1).

| $\alpha$ | $\beta$ |
|---|---|
| 0.25 | 0.99 |
| 0.5 | 0.91 |
| 0.75 | 0.76 |
| 1.0 | 0.60 |
| 2.0 | 0.26 |

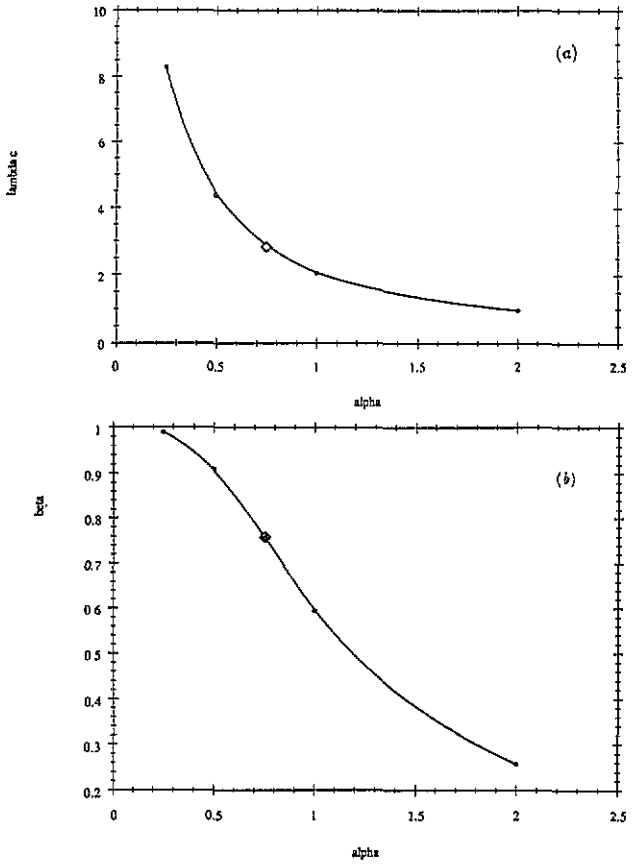


**Figure 1.** (*a*) $\lambda_c(\alpha)$ as obtained from CAM. (*b*) $\beta(\alpha)$ as obtained from CAM. The result of computer simulations for $\alpha = 0.75$ is represented by $\Diamond$.

free parameters. In table 3 we list the values of $\lambda_c$ and $\omega$ obtained in this manner.

The exponent $\beta$ was then estimated by fitting $A_L$ to the form (1); better results were indeed obtained by considering also higher-order corrections. The results are listed in table 4.

In figure 1(*a*) and 1(*b*) we have plotted $\lambda_c(\alpha)$ and $\beta(\alpha)$ as obtained from the above CAM estimates. Improvement of the results is expected if bigger clusters are considered.

### *3.2. Results obtained from approximation B*

The implementation of this approximation is more demanding on computer time, therefore we have restricted our study to just one case: $\alpha = 2.0$. Fits were then obtained using the procedure described above in subsection 3.1.

In table 5 we list the values of $\lambda_c^L$ and $A_L$, for cluster sizes up to $L = 9$. The value of $\omega$ obtained within this approximation is 0.91, very close to the best estimates for $1/\nu_\perp$ in the short-range case; however the values of $\lambda_c$ and $\beta$ obtained here—0.977 and 0.31, respectivel—are slightly above the estimates of 3.1 for $\alpha = 2.0$. This may be a consequence of lower precision in the estimates for $L = 8, 9$; bigger cluster sizes were not considered due to computer time limitations.

**Table 5.** $\lambda_c^L$ and $A_L$ for different cluster sizes obtained using approximation B (subsection 2.3.2) for $\alpha = 2.0$.

| $L$ | $\lambda_c^L$ | $A_L$ |
|---|---|---|
| 3 | 1.365 8157 | 1.297 328 |
| 4 | 1.263 0742 | 1.660 435 |
| 5 | 1.204 0731 | 1.989 729 |
| 6 | 1.165 7804 | 2.293 197 |
| 7 | 1.138 8986 | 2.576 217 |
| 8 | 1.118 9671 | 2.842 41 |
| 9 | 1.103 5808 | 3.094 91 |

## 4. Results of computer simulations

As mentioned before, computer simulations for these long-range systems are extremely demanding on computer time. Steady-state simulations of short-range non-equilibrium systems close to an absorbing state are already very costly because of critical slowing down and the fact that small systems enter quickly the absorbing state even in the supercritical region, thus requiring the observation of large systems for a long time; and, of course, one needs to average over a large number of samples to obtain a good statistics.

Just for comparison and for an awareness of these limitations, we show here some preliminary results of a computer simulation done on a system of size $L = 1000$, for $\alpha = 0.75$. The size of the system prevented us from obtainig results very close to the critical point, as this would require a very great number of realizations to allow for a few not to enter the absorbing state.

In figure 2 we show a log–log plot of the steady concentration of vacant sites $\rho$ versus $\lambda_c - \lambda$ for $\lambda_c = 2.825$. The slope of the line is $\beta = 0.758$ . For comparison we have also displayed these simulation results for $\lambda_c$ and $\beta$ in figure 1($a$) and ($b$).

To obtain a better statistics on bigger systems and for different values of $\alpha$ would certainly be desirable. However this is certainly a difficult task when one considers that 20 realizations of this system for one value of $\lambda$ took about 240 hr of CPU time on a

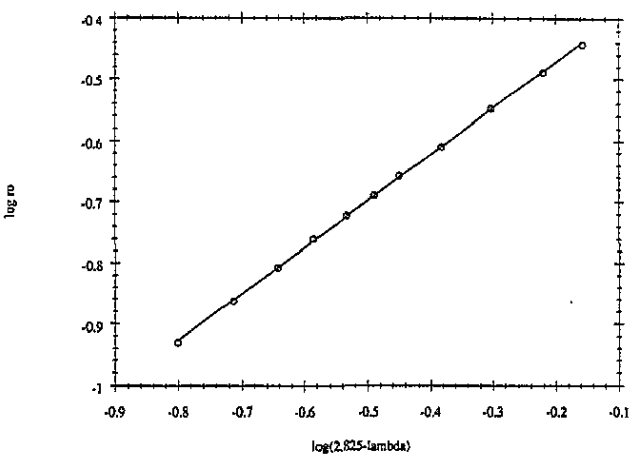


**Figure 2.** Log–log plot of the order parameter versus $\lambda_c - \lambda$ , as obtained from computer simulations of a system of size $L = 1000$.

HP720 workstation.

## 5. Discussion and concluding remarks

The use of CAM has enabled us to study the $\alpha$ -dependence of the critical parameter $\lambda_c$ and the critical exponent $\beta$ in a long-range, non-equilibrium system, for which other methods used in short-range, non-equilibrium systems face great difficulties.

According to field theory [5], the upper critical dimensionality is $d_c = 2\alpha$, so the dimensionality $d = 1$ of our system is above $d_c$ for $\alpha < 0.5$, and the mean-field critical exponent $\beta_{MF} = 1$ should be expected in this case; on the other hand, it is predicted that the short-range critical behaviour should be recovered for $\alpha \geqslant 2$.

Within approximation A, we obtain $\beta = 0.990$ for $\alpha = 0.25$ and $\beta = 0.910$ for $\alpha = 0.5$ (where some logarithmic corrections might be expected, since $d = d_c$); so the results seem quite reasonable and are likely to be improved if bigger clusters are used.

As for the case of $\alpha = 2.0$, we obtain $\beta = 0.26$ for approximation A; consideration of larger cluster sizes is expected to increase the estimate, and therefore this is consistent with the value $\beta = 0.277$ for the (short-range) contact process.

The analysis of the results for approximation B poses some questions. Indeed, the best fit for $\lambda_c$ not only is slightly above the value obtained from approximation A, but also when used to obtain $\beta$, it gives estimates above the short-range value $\beta = 0.277$. However, a slightly lower value of $\lambda_c$ already gives estimates consistent with the short-range value; this difficulty may be eliminated if more computer time is dispended in attaining higher accuracy. Interestingly, even though the reason might not be very clear at the moment, this approximation gives an estimate for $\omega$ consistent with the short-range value for $1/\nu_\perp$. Further study is required to clarify these points.

One clear advantage of this approach is the fact that one is not limited by small $\epsilon = 2\alpha - 1$, as in the first-order $\epsilon$-expansion expressions for the critical exponents.

## Acknowledgments

This work was partially supported by JNICT, project CEN-386/90. ALF thanks INIC Portugal for financial support.

## References

[1] Dickman R 1993 *Int. J. Mod. Phys.* C **4** 271 and references therein
[2] Marques M C 1990 *Physica* **163A** 915
[3] Dyson F J 1969 *Commun. Math. Phys.* **12** 91
[4] Janssen H K 1981 *Z. Phys.* B **42** 151
[5] Grassberger P 1982 *Z. Phys.* B **47**365
[6] Suzuki M 1986 *Fractals in Physics* p 273; *J. Phys. Soc. Japan* **55** 4205
[7] Konno N and Katori M 1990 *J. Phys. Soc. Japan* **59** 1581
[8] Ferreira A L C and Mendiratta S K 1993 *J. Phys. A: Math. Gen.* **26** L145–50
[9] Bergensen B and Rácz Z 1991 *Phys. Rev. Lett.* **67** 3047
[10] Monroe J L, Lucente R and Hourland J P 1990 *J. Phys. A: Math. Gen.* **23** 2555
[11] Suzuki M 1986 *Phys. Lett.* **116A** 375
[12] Lipowski A and Suzuki M 1992 *Preprint* University of Tokyo
[13] ben-Avraham D and Kohler J 1992 *Phys. Rev.* A **45** 8358
[14] Barber M N 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (New York: Academic) pp 145–226